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On Toda equation and half BPS supergravity solution in M-theory

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Abstract: Recently, it was shown that half BPS Supergravity solution of theories with SU(2|4) symmetry algebra is given uniformly by determining a single function which obeys three dimensional continuous Toda equation. In this paper, we study the scale invariant solution of Toda equation. Our motivation is that some solutions of half BPS sector of IIB supergravity, as one excepts from the fermion description of the theory, are scale invariant. By defining two auxiliary functions we prove that such solutions of Toda equation obey cubic algebraic equation. We obtain some simpl solutions of Toda equation specially, we observe that the PP-wave solution can be written in this fashion.

Keywords: M-Theory, AdS-CFT Correspondence.

Contents

1. Introduction

The quantum theory of space-time has been one of the most important and basic concepts in physic for long time. String theory, as a good candidate for theory of every thing, opens different wonderful windows to the theory of gravity. Holography [1] is one of the most important and beautiful subject which is statement about duality between string theory and gauge theory. AdS/CFT correspondence [2] is a concrete example of such duality which is a strong/weak duality. It has been found that there is also a weak/weak correspondence due to large quantum number limit(BMN sector) in gauge theory and semi-classical strings [3].

The half BPS sector of such theories has some important property which helps one to study the AdS/CFT duality in different regime. In fact, Information coming from BPS sector is protected by supersymmetry and so all computations can reliable either at weak coupling or strong coupling.

Recently [4, 5], the dual super gravity solutions of half BPS sector of theories with $PSU(2,2|4)$ SUSY algebra was found in a uniform way. All such theories are classified by different subgroup of $PSU(2,2|4)$ with 16 supercharges which are $SU(2,2|2)$, $PSU(2|2) \times PSU(2|2) \times U(1), SU(2|4)$ and SuperPoincare part of $PSU(2,2|4)$.

The supergravity solutions are demanded to have some specific properties. More precisely,

i) solutions should have globally well-defined time-like (or light-like) Killing vector isometry,

ii) the bosonic part of the isometries should be compact,

iii) solutions should be smooth.

Interestingly, it was shown in [4, 8] that supergravity theory with PSU(2|2)$\times PS-$U(2|2)$\times$ U(1) algebra, which arises in IIB theory naturally, is dual to a free fermion description of $\mathcal{N}=4$ super Yang-Mills on $R\times S^3$. The phase space of two theories is given by two dimensional space with two topologically different region. The solution for different boundary condition gives different deformation of maximally supersymmetric $AdS_5\times S^5$ space. There have been done a lot of work after that for understanding these duality better [9].

It was also shown that [5] different gravity solution with SU(2|4) SUSY algebra can be dual to different theories including: Plane wave Matrix model(BMN Matrix model), $2+1$ super Yang-Mills on $R\times S^2$ and $\mathcal{N}=4$ super Yang-Mills on $R\times S^3/Z_k$. In fact, this solutions which appear in M-theory are all deformation of $AdS_7\times S^4$ and $AdS_4\times S^7$ spaces. Finding a free fermion description is an important aim in completing such duality.

In this paper, at next section, we review the LLM geometry in IIB supergravity and discuss some basic points about the phase space of solutions. We observe that some interesting solutions are invariant under scaling. Then, we introduce the LLM geometry in M-theory which can be determined by solution of Toda equation. In section 3 we study the scale invariant solutions of Toda equation and prove that such solutions obey a cubic algebraic equation. In section 4 we find some simple scale invariant solutions so called "separable" solution, PP-wave solution and M-5 brane solution. We discuss the dual gauge theory of these solutions briefly.

2. Review of LLM geometry

At two next subsection we review briefly the LLM geometry arising in theories with PSU(2|2)×pSU(2|2)× U(1) or SU(2|4) superalgebra. The former case gives the supergravity solution for half BPS sector in IIB theory and the later, solution for half BPS sector in M-theory.

2.1 LLM geometry in IIB theory

For the case PSU(2|2)$\times$PSU(2|2)$\times$U(1) algebra, the bosonic part is SO(4)$\times$SO(4)$\times$U(1). Considering the whole requirements about gravity solution, the solution was found as [4]

$$\begin{aligned} ds^2 &= -h^{-2}(gdt+V_i dx^i)^2 + h^2(dy^2+dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G}d\tilde{\Omega}_3^2, \\ h^{-2} &= y\cosh G, \qquad y\partial_y V_i = \epsilon_{ij}\partial_j z, \\ z &= \frac{1}{2}\tanh G \end{aligned} \tag{2.1}$$

Here, the dilaton and axion field were assumed to be constant and the three-form field strengths are zero. The local coordinate y is defined using a closed one from constructed by spinor bilinears. It has special property since is the product of the radii of two spheres. Whole solution can be determined by a single function z which satisfies following differential equation

$$\partial_i \partial_i z + y \partial_y \left(\frac{\partial_y z}{y} \right) = 0. \tag{2.2}$$

Using the change of variable as $\Phi = z/y^2$, this differential equation (2.2) can be reduced to a six dimensional Laplace equation with spherical symmetry in four of the dimensions, y is then the radial variable in these four dimensions. At $y = 0$ the product of the radii of two spheres is zero. So one could have singularities at $y = 0$ unless z has a special behavior. As it was shown in, the smoothness condition implies that we have two boundary condition on $x_1 x_2$ plain.

$$z = \frac{1}{2} \quad S^3 \text{ shrinks}, \qquad z = -\frac{1}{2} \quad \tilde{S}^3 \text{ shrinks} \tag{2.3}$$

So, the moduli space of solution can be determined by specifying regions on $x_1 x_2$ plain that either $z = \frac{1}{2}$ or $z = -\frac{1}{2}$ (which so called black region or white region respectively). More interestingly, an arbitrary configuration of two different regions corresponds to phase space of free fermions in an specific gauge theory. In fact, the author of has shown that the half BPS sector of the $\mathcal{N} = 4\,\mathrm{U}(N)$ SYM on $R \times S^3$ is equivalent to an N fermion system in one dimensional harmonic oscillator potential.

An important observation which has been done in LLM paper is that the flux of field strengths either for the five form or dual five form field is proportional to the area of regions where S^3 or $\tilde{S}^3$ shrinks. Using the corresponding fermion phase space quantization, one can write the precise quantization condition on the area of the droplets in the $x_1 x_2$ plane as

$$(Area) = 4\pi^2 l_P^4 N \qquad \text{or} \qquad \hbar = 2\pi l_p^4, \tag{2.4}$$

which means that we have a fundamental length in theory corresponding to each branes [4].

At the level of supergravity solution, such property means that after a scaling of coordinates x_i, y such that $x_i \to \lambda x_i, y \to \lambda y$ then we have to have $ds^2 \to \lambda ds^2$ [4, 10]. Here λ is an arbitrary constant and we actually perform rigid deformation on the shape of the original configuration. Such scaling behavior is important because of its relation to Penrose limit [10].

Now we focus on finding solutions which are invariant under such scaling namely

$$z(\lambda x_i, \lambda y), = z(x_i, y). \tag{2.5}$$

Such solutions are a particular subset of solutions of (2.2) ,specially when we have multi boundary conditions we can't write the solution in this way uniformly.

Using the variables $\eta = \frac{x_1}{y}$ and $\zeta = \frac{x_2}{y}$ the equation (2.2) reduces to

$$(1 + \eta^2)\partial_\eta^2 z + (1 + \zeta^2)\partial_\zeta^2 z + 2\eta\zeta\partial_{\eta\zeta}^2 z + 3\eta\partial_\eta z + 3\zeta\partial_\zeta z = 0. \tag{2.6}$$

One can find different kind of solutions of (2.2) for example separable solution where $z(\eta,\zeta) = f(\eta) + g(\zeta)$. In this case one obtains

$$\begin{aligned} f(\eta) &= k_1 + \frac{k_1'}{\sqrt{1+\eta^2}} - \frac{K}{4}\ln(1+\eta^2) - \frac{K}{2}\int \frac{\sinh^{-1}\eta}{(1+\eta^2)^{3/2}}d\eta, \\ g(\zeta) &= k_2 + \frac{k_2'}{\sqrt{1+\zeta^2}} + \frac{K}{4}\ln(1+\zeta^2) + \frac{K}{2}\int \frac{\sinh^{-1}\zeta}{(1+\zeta^2)^{3/2}}d\zeta, \end{aligned} \tag{2.7}$$

where k_1, k_1', k_2, k_2' and K are arbitrary constants.

For the case where we have an isometry on x_1 direction one has to set $K = 0$ and the above differential equation has following solution

$$z(\eta) = k_1 + k_2' \frac{\eta}{\sqrt{1+\eta^2}}, \tag{2.8}$$

which is the pp-wave solution [11]. From the boundary condition (2.3) we have $k_1 = 0$ and $k_1' = \frac{1}{2}$. This pp-wave solution has two different boundary where extended to infinity. In fact, such extension allows us to write the solution in terms of η, ζ variables.

Motivated by the above construction of solutions by η, ζ variables, we want to study the supergravity solutions in M-theory.

2.2 The LLM geometry in M-theory

The other class of subgroup of PSU(2, 2|4) with 16 supercharges is SU(2|4). The bosonic part of the symmetry is SO(6) × SO(3) × U(1) and the 11 dimensional supersymmetric solutions with that symmetry structure is given by [4]

$$\begin{aligned} ds^2_{11} &= -4e^{2\lambda}\Big[(1+y^2e^{-6\lambda})(dt+V_idx^i)^2 + \frac{e^{-6\lambda}}{1+y^2e^{-6\lambda}}[dy^2 + e^D(dx_1^2+dx_2^2)] + \\ &\qquad +d\Omega_5^2 + y^2e^{-6\lambda}d\tilde{\Omega}_2^2\Big] \\ e^{-6\lambda} &= \frac{\partial_y D}{y(1-y\partial_y D)}, \qquad\qquad V_i = \frac{1}{2}\epsilon_{ij}\partial_j D \end{aligned} \tag{2.9}$$

The function D determines the whole solution and obeys three dimensional continuous version of the Toda equation

$$\partial_{x_1}^2 D + \partial_{x_2}^2 D + \partial_y^2 e^D = 0. \tag{2.10}$$

The boundary condition for having non singular solution in LLM construction are that at $y = 0$

$$\begin{aligned} \partial_y D = 0, \qquad D &= \text{finite}, \qquad S^2 \quad \text{shrinks} \\ D \sim \ln y \qquad & \qquad\qquad\quad\;\; S^5 \quad \text{shrinks} \end{aligned} \tag{2.11}$$

These conditions ensure that the y coordinate combines with the sphere coordinates in a non singular fashion.

The moduli space of solutions again is a two dimensional space with different configuration of black (S^2 shrinking sphere) or white (S^5 shrinking sphere) regions.

However, the existence of a free fermion description is not very clear. In fact, the fluxes of four form field strength and its dual is given by

$$N_5 \sim \int_{\mathcal{D}} dx_1 dx_2 2(y^{-1} e^D)|_{y=0}, \qquad N_2 \sim \int_{\mathcal{D}} dx_1 dx_2 2(e^D)|_{y=0}. \tag{2.12}$$

In both cases, the fluxes are given by the area measured with the metric obtained from D. So, one has to find the solution of Toda equation (2.10) at first, and then, computes the number of 2-brane or 5- brane. Furthermore, from the nonlinearity of Toda equation, solutions of such differential equation has not a well behavior under scaling of coordinates. For example, see the PP-wave solution (4.4). But we have an important property of solution. The three dimensional Toda equation has SU(∞) (conformal) symmetry in x_1x_2 plane in which the form of the metric (2.9) is preserved under such symmetry. Defining $z = x_1 + ix_2$ the symmetry is

$$z \to f(z), \qquad D \to D - \log(|\partial f|^2), \tag{2.13}$$

where $f(z)$ is an arbitrary holomorphic function of z. So, one may hope that by a conformal transformation obtains a solution which has well behavior under scaling. At the end, the uniqueness of the solutions of *nonlinear* Toda equation with boundary conditions (2.11) is another issue which one has to consider it. For example, for the pp-wave solution (4.5) in x_1x_2 plane we have three nonequal solution for cubic algebraic equation. Defining

$$S_\pm = \Big(\frac{y^2}{4} + \frac{x^3}{27} \pm \sqrt{\frac{y^4}{16} + \frac{y^2x^3}{54}}\Big)^{1/3}, \tag{2.14}$$

then equation (4.5) has the following solutions

$$\begin{aligned}
(e^D)_1 &= S_+ + S_- + \frac{x}{3}\\
(e^D)_2 &= -\frac{1}{2}(S_+ + S_-) + \frac{1}{2}i\sqrt{3}(S_+ - S_-) + \frac{x}{3}\\
(e^D)_3 &= -\frac{1}{2}(S_+ + S_-) - \frac{1}{2}i\sqrt{3}(S_+ - S_-) + \frac{x}{3}.
\end{aligned} \tag{2.15}$$

$(e^D)_1$ satisfies the S^2 *shrinking* boundary condition for $x \geq 0$ and $(e^D)_2$ or $(e^D)_3$ (which are nonequal) satisfy the S^5 *shrinking* boundary condition for $x \leq 0$. The flux driven from $(e^D)_2$ or $(e^D)_3$ are equal up to a minus sign. Interestingly, in $\rho\theta$ coordinate (which will be defined in section 4.2), we have a unique solution. Such behavior also exist for $AdS_4 \times S^7$ or $AdS_7 \times S^4$ solutions [4] where in (x_1, x_2, y) coordinate we have a quartic algebraic equation. Considering all above difficulties cause some ambiguities for having a fermion description at CFT side.

Motivated by similar case in IIB theory, we will study some solutions of Toda equation in terms of rational variables. These solutions, at least, have well behavior under scaling of coordinates.

Note that if we find a solution D such that $D(\lambda x_i, \lambda y) = D(x_i, y)$, then at the level of metric (2.9) we have

$$(x_i, y) \mapsto (\lambda x_i, \lambda y) \Rightarrow \quad ds^2 \mapsto \lambda^{2/3} ds^2. \tag{2.16}$$

such behavior comes from the fact that the coordinate y has dimension $(length)^3$.

3. Scale invariant solution of Toda equation

Three dimensional Toda equation is a limit of the exactly solvable Toda molecule equation and appears in a variety of physical cases, running from the theory of hamiltonian systems to general relativity in the theory of self dual Einstein spaces or in the problem of finding four dimensional hyper-Kaheler manifold with a rotational Killing vector [12].

Unfortunately, finding exact solution of the Toda equation is very hard and only few solutions are known. There are also some methods based on group theoretical consideration in which one can find the symmetry structure of the equation and corresponding generator and then it is possible to reduces the equation to a simpler equation. The Toda equation allows an $infinite$ dimensional symmetry algebra, a realization which is given by generators obeying Virasoro algebra without central charges(Witt algebra). The reduced equation in this way gives rise to instanton solutions. For example, one may consider the separable solution in the sense that $D(x_i, y) = F(x_i) + G(y)$, which cases that the Toda equation reduces to Liouvile equation which has well known instanton solution.

For the case that we have an additional isometry on x_1 direction one can use the following change of variables [13]

$$e^D = \rho^2, \quad y = \rho\partial_\rho V, \quad x = \partial_\theta V, \tag{3.1}$$

and reduces the Toda equation to a three dimensional Laplace equation with cylindrical symmetry as

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho V) + \partial_\theta^2 V = 0 \tag{3.2}$$

Considering the boundary condition in $\rho\theta$ plane one realizes that [5] the problem in this plane reduces to finding solution for an electroestatic Laplace equation with some conducting disk located at some constant θ_i . The solution is determined by specifying the charge of the disks which is proportional to M_2 brane number and distance between disks which is proportional to M_5 brane number. Even in this case the exact solution for different configuration of disks is not known and few solutions were obtained.

In the procedure which we will discuss it, we derive some of solutions such that have scale invariance property and relate to solution in electrostatic problem. Interestingly, we will see that all solutions obtained in this fashion obey a cubic algebraic equation.

The three dimensional Toda equation (2.10) using change of variables

$$\eta = x_1/y, \qquad \zeta = x_2/y; \tag{3.3}$$

reduces to

$$\partial_\eta^2 D + \partial_\zeta^2 D + \eta^2\partial_\eta^2 e^D + \zeta^2\partial_\zeta^2 e^D + 2\eta\partial_\eta e^D + 2\zeta\partial_\zeta e^D + 2\eta\zeta\partial_{\eta\zeta}^2 e^D = 0. \tag{3.4}$$

Defining the auxiliary functions $U(\eta,\zeta)$ and $V(\eta,\zeta)$ (in which $\partial_\zeta U$ and $\partial_\eta V$ are not zero) as

$$U(\eta,\zeta) = \partial_\eta D + \eta^2\partial_\eta e^D,$$

$$V(\eta,\zeta) = \partial_\zeta D + \zeta^2 \partial_\zeta e^D, \tag{3.5}$$

and using

$$\begin{aligned}
\partial^2_{\eta\zeta} e^D &= e^D(\partial_\eta D \partial_\zeta D + \partial^2_{\eta\zeta} D) = \frac{\partial_\zeta U - \partial_\eta V}{\eta^2 - \zeta^2} \\
\partial^2_{\eta\zeta} D &= \frac{1}{2}(\partial_\zeta U + \partial_\eta V) - \frac{1}{2}\frac{\eta^2+\zeta^2}{\eta^2-\zeta^2}(\partial_\zeta U - \partial_\eta V),
\end{aligned} \tag{3.6}$$

after some computations one finds

$$(e^D)^3 + f(\eta,\zeta)(e^D)^2 + g(\eta,\zeta)(e^D) + h(\eta,\zeta) = 0, \tag{3.7}$$

where

$$\begin{aligned}
f(\eta,\zeta) &= \frac{\eta^2(\eta^2+2\zeta^2)\partial_\eta V - \zeta^2(\zeta^2+2\eta^2)\partial_\zeta U}{(\eta^2\zeta^2)(\eta^2\partial_\eta V - \zeta^2\partial_\zeta U)} \\
g(\eta,\zeta) &= \frac{(\eta^2-\zeta^2)UV + (2\eta^2+\zeta^2)\partial_\eta V - (2\zeta^2+\eta^2)\partial_\zeta U}{(\eta^2\zeta^2)(\eta^2\partial_\eta V - \zeta^2\partial_\zeta U)} \\
h(\eta,\zeta) &= \frac{\partial_\eta V - \partial_\zeta U}{(\eta^2\zeta^2)(\eta^2\partial_\eta V - \zeta^2\partial_\zeta U)}.
\end{aligned} \tag{3.8}$$

and the Toda equations (2.10) reads

$$\partial_\eta U + 2\frac{\eta\zeta}{\eta^2-\zeta^2}\partial_\zeta U(\eta,\zeta) + \partial_\zeta V - 2\frac{\eta\zeta}{\eta^2-\zeta^2}\partial_\eta V(\eta,\zeta) = 0. \tag{3.9}$$

This is a first order linear partial differential equation. One can write various solutions for equation (3.9) such

$$\begin{aligned}
&\text{i)} \quad U(\eta,\zeta) = F_1\left(\frac{\zeta}{\eta^2+\zeta^2}\right), \qquad V(\eta,\zeta) = F_2\left(\frac{\eta}{\eta^2+\zeta^2}\right) \\
&\text{ii)} \quad U(\eta,\zeta) = \frac{b}{a}V(\eta,\zeta) = F_3\left(\frac{a\eta+b\zeta}{\eta^2+\zeta^2}\right),
\end{aligned} \tag{3.10}$$

where F_i's are general functions. Notice that the functions F_1 and F_2 are the background solutions for equation (3.9). From the linearity of equation (3.9) the superposition of any solutions of this equation is also a solution of (3.9). But, one has to check the consistency condition in which the obtained solution (3.7), (3.9) should also satisfy equations (3.5). This consistency check is hard and we only present some simple solutions using somehow different idea.

4. Generating some solutions

Solving the Toda equation even in the form (3.5), (3.7) and (3.9) is very hard and the equation (3.7) presents an special property of scale invariant solution. In fact, one has to check the consistency conditions of solutions. In the following subsections, we present some simple cases known as separable solution and next to that we find PP-Wave solution in term of these scale invariant variables. At the next section, we drive a solution in terms of η only by solving the Toda equation directly.

4.1 Separable solution

The simplest solution can be obtained by rewriting the Toda equation in terms of $(z, \bar{z}, y)$ variables and assuming that $e^D = F(\tilde{\eta})G(\tilde{\zeta})$ where $\tilde{\eta} = \frac{z}{y-y_0}$ and $\tilde{\zeta} = \frac{\bar{z}}{y-y_0}$. In this case the solution is

$$e^D = (y - y_0)\left(\frac{\bar{z}^{m+1}}{z^m}\right) \qquad \text{or} \qquad e^D = (y - y_0)\left(\frac{z^m}{\bar{z}^{m+1}}\right) \tag{4.1}$$

Using conformal transformation (2.13) and the reality condition on D one obtains

$$e^D = c_1 y + c_2. \tag{4.2}$$

This solution doesn't preserve the S^2*shrinking* boundary condition and so is a singular solution.

In electrostatic point of view, we have a configuration with a line of charge at $\rho = 0$ axis in the presence of the external potential V_b and potential V as [5]

$$V = -\frac{\pi N}{2k}\log\rho + V_b, \qquad V_b = \frac{1}{g_s k}(\rho^2 - 2\eta^2). \tag{4.3}$$

After compactifying the x_1 direction one obtains a supergravity solution corresponding to $\mathcal{N} = 4$ super Yang-Mills on $R \times S^3/Z_k$ [5, 14]. This theory is an orbifold of $\mathcal{N} = 4$ SYM which the simplest dual orbifolded supergravity solution is $AdS_5/Z_k \times S^5$. The singularity at $y = 0$ corresponds to the Z_k orbifold fixed points in IIB sense.

As it was shown in, this solution can be viewed as solution for the near horizon geometry of semilocalised intersecting M2-branes [15]. It can also be dual to a superconformal theory, since it is a AdS_2 fibration [5, 16].

4.2 PP wave solution

Using changes of variables (3.1), the Toda equation reduces to a cylindrically symmetric Laplace equation in three dimensions(3.2) which has a polynomial solution as

$$V = \rho^2\eta - \frac{2}{3}\theta^3 \tag{4.4}$$

The boundary in this case transformed to $\rho = 0$ and $\theta = 0$. Solution preserves S^5 boundary condition at $\rho = 0$ and S^2boundary condition at $\theta = 0$. So from electrostatic point of view, we have an infinite disk at $\theta = 0$ and only $\theta \geq 0$ is physically meaningful.

By the above changes of variables one finds the function D obeys a cubic algebraic equation as

$$(e^D)^3 - x(e^D)^2 - \frac{y^2}{2} = 0 \tag{4.5}$$

Now acting a conformal transformation as $z' = (z)^{3/2}$, one can rewrite the above equation as

$$(e^{D'})^3 - \frac{2}{9}\left(\left(\frac{\tilde{\eta}}{\tilde{\zeta}}\right)^{1/2} + \left(\frac{\tilde{\zeta}}{\tilde{\eta}}\right)^{1/2}\right)(e^{D'})^2 - \frac{32}{729}\frac{1}{\tilde{\eta}\tilde{\zeta}} = 0, \tag{4.6}$$

where $\tilde{\eta} = \frac{z}{y}$ and $\tilde{\zeta} = \frac{\bar{z}}{y}$. So we find a solution of (3.4) in terms of $\tilde{\eta}\tilde{\zeta}$ variables such that

$$f = -\frac{2}{9}\left(\left(\frac{\tilde{\eta}}{\tilde{\zeta}}\right)^{1/2} + \left(\frac{\tilde{\zeta}}{\tilde{\eta}}\right)^{1/2}\right), \qquad g = 0, \qquad h = -\frac{32}{729}\frac{1}{\tilde{\eta}\tilde{\zeta}}. \tag{4.7}$$

The solution (4.4), (4.6) is pp-wave solution in M-theory with particle by nonzero $-p_-$ which are translationally invariant along x_-. After compactifying x_- direction one finds dual gravity solution corresponding to the plane wave (BMN) matrix model [3, 4]. In fact in IIA variables the solution in the UV region goes over to the UV region of the solution for D_0 branes [5, 6, 17].

For rotationally invariant solutions in x_1x_2 plane, one may perform a conformal transformation to map circular droplet to strips. In fact, the plane and cylinder can be mapped into each other conformally. In this case writing two dimensional metric $dx_1^2 + dx_2^2$ in polar coordinates (r, θ) and using change of variables as $x_2 \to \ln r$ and $D \to D + 2\ln x_2$, one obtains the two dimensional Toda equation which has a scale invariant solution after performing another conformal transformation $z' = (z)^{3/2}$.

4.3 M-5 brane solution

In this section, we consider the case that D is not a function of x_1 and we want to find the scale invariant solution of Toda equation by solving the differential equation directly. We will also consider $x_2 = x$. Then, using $\eta = \frac{x-x_0}{y-y_0}$ the Toda equation can be written as

$$\partial_\eta(D) + \eta^2\partial_\eta(e^D) = 2c, \tag{4.8}$$

where c is an arbitrary constant. Defining $e^D = \rho^2$ one may rewrite this equation as

$$\frac{\partial\eta}{\partial\rho} = \frac{\rho}{c}\eta^2 + \frac{1}{\rho c}. \tag{4.9}$$

The solution can be obtained by using the Ricatti change of variable

$$\eta(\rho) = -\frac{c}{\rho}\frac{W'(\rho)}{W(\rho)}. \tag{4.10}$$

Then

$$W''(\rho) - \frac{1}{\rho}W'(\rho) + \frac{1}{c^2}W(\rho) = 0. \tag{4.11}$$

One can easily find that the solution of this equation has the following general form,

$$W(\rho) = c_1\rho J_1\left(\frac{\rho}{c}\right) + c_2\rho Y_1\left(\frac{\rho}{c}\right). \tag{4.12}$$

where J_1 and Y_1 are the first and second Bessel functions respectively. For the case $c_2 = 0$ using (4.10) we obtain

$$\eta(\rho) = -\frac{1}{\rho}\frac{J_0(\frac{\rho}{c})}{J_1(\frac{\rho}{c})} \tag{4.13}$$

Although one would like to find an explicit expression for inverse function $e^D = \rho^2(\eta)$, but we are interested in analyzing the solution and boundary condition in the ρ coordinate.

For the case where the S^5 shrinks one needs as $y \to y_0$ then $\eta \sim \frac{1}{\rho^2}$. Since when $\rho \to 0$, $J_0(\frac{\rho}{c}) \to 1$ and $J_1(\frac{\rho}{c}) \to \frac{\rho}{2c}$, the boundary condition is satisfied.

For the case where S^2 shrinks using $\frac{\partial D}{\partial y} = \frac{2}{\rho}\frac{\partial \rho}{\partial y}$ one finds

$$\frac{\partial D}{\partial y} = \frac{-2c(x-x_0)}{(y-y_0)^2 + \rho^2(x-x_0)^2}. \tag{4.14}$$

We see that for $y_0 = 0$ the expression(4.14) will be zero only at $x = \infty$ and for $y_0 \neq 0$ at $x = x_0$ and $x = \infty$. However, solution(4.13) shows that ∞ corresponds to $\rho = 0$ which means that D is not finite. So the boundary condition doesn't preserve at ∞. For the other case, one has $\rho \neq 0$ and so the second boundary condition preserved at $x = x_0$. One may consider (4.13) as a solution which produces $AdS_5 \times X$ space which X is a six dimensional compact space. In fact, after an analytic continuation of original Supergravity solution (2.9) one finds that this $AdS_5 \times X$ solutions determined with a single function D which obeys the same Toda equation but it should preserves following boundary conditions [4, 18]

$$\begin{aligned} \partial_y D = 0, \qquad & D = \text{finite}, \quad S^2 \quad \text{shrinks} \\ D \sim \ln(y-y_0) \qquad & y_0 \neq 0 \qquad\quad S^5 \quad \text{shrinks}. \end{aligned} \tag{4.15}$$

One may also consider $c = 0$ at (4.13), but this does not imply an interesting solution for the metric.

Let us consider the case $y_0 = 0$ and $x_0 = 0$ and $c = \pm 1$. In fact noticing that if $D(x, y)$ be a solution of Toda equation then $D(x_i, \lambda y) + 2\ln y$ is also a solution of Toda equation, then, one can generate the solution for the case that $c \neq \pm 1$ using (4.13).

From the electrostatic point of view, the above solution corresponds to a potential $V = J_0(\frac{\rho}{c})e^{\frac{\eta}{c}}$. Both two cases $c = \pm 1$ imply a singular solution but considering the linearity of Laplace equation one can write a regular solution as $J_0(\rho)\sinh\eta$. Obtaining a regular solution in which the change of variables (3.1) be well defined, imposes choosing following solution [5]

$$V(\rho, \eta) = I_0(\rho)\sin(\eta), \tag{4.16}$$

where $I_0(\rho)$ is the first modified Bessel function. This can be done by choosing $c = i$, noticing the fact that only when a regular solution is at hand one can choose an imaginary value for c. This solution corresponds to two infinite separated disks in $\rho\theta$ plane.

The solution (4.16) in IIA language is dual to little string theory on $R \times S^5$. In large ρ regime the solution asymptotes to N IIA NS_5 branes wrapping on $R \times S^5$ [19].

5. Conclusion

The free fermion description of $\mathcal{N} = 4$ SYM on $R \times S^3$ has had an important rule in AdS/CFT correspondence [8]. In fact, it was shown that the moduli space of half BPS sector of IIB supergarivity solutions exactly mapped to phase space of fermion system [4, 5].

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This space divided to two regions where either S^3 sphere shrink or $\tilde{S}^3$. Considering the fact that the area of such regions gives the number of brane and considering similar description in fermion phase space side, one realize that the whole solution is invariant under scaling although, a generic solution may not be scale invariant.

Motivated by such observation, we study some scale invariant solutions in half BPS sector of supergravity solutions in M-theory, although the existence of a free fermion description has not been understood yet. By introducing two auxiliary functions, we proved that all such solutions can be obtained by solving a cubic equation.

Unfortunately, even with this simplification, finding the solution is hard, because that one has to do a consistency check on solution. We obtain a simple solution so called "separable" solution which in IIA sense corresponds to solutions of little string theory. We also write the PP-wave solution in terms of this homogeneous coordinates. Finally, when we have an addition isometry, we find the solution of Toda equation directly. Using this solution one can find a regular solution in electrostatic point of view.

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